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Solution of the boundary value problem for the integrable discrete SRS system on the semi-line*

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Abstract. The complete solution of the initial-boundary value problem for the integrable discrete version of the stimulated Raman scattering system on the semi-line is constructed by means of the inverse spectral transform. The spectral data obey a Riccati time evolution equation which allows for soliton generation out of a medium initially at rest. It is also proved that the construction of the solution at any finite distance actually results in solving an algebraic system.

1. Introduction

Generalities

The method of the inverse spectral transform (IST) was originally developed for partial differential equations in continuous variables associated with an initial value problem. These are, for instance, the Korteveg de Vries equation [1], the nonlinear Schrödinger equation [2], the sine–Gordon equation [3, 4], etc. Very soon after that the theory was also constructed for discrete nonlinear evolutions (continuous time) such as the Toda lattice and the Ablowitz–Ladik equation [5]. For a review see, for instance, [6].

Soon after the discovery of the integrability of all these physically interesting nonlinear evolution equations, another class of nonlinear problems were shown to share some of the integrability properties. The first instance was the self-induced transparency (SIT) equations of McCall and Hahn [7] for which the Lax pair has been found by Lamb [8] and which have been solved completely by Ablowitz *et al* [4]. The first essential property of such a system is to have a dispersion relation which is a non-analytic function of k (the wavenumber), and this can be settled in a very general formalism [9, 10], also for $(2+1)$-dimensional problems [11].

The second fundamental property of SIT is that its solution can be extended to an arbitrary initial state of the medium, allowing then the solution of an initial/boundary value problem [12], which is of interest for superfluorescence. This property has then been generalized to the class of evolutions with singular dispersion relations related to the Zakharov–Shabat spectral problem in [13].

Among this class, there is the stimulated Raman scattering (SRS) system, first shown to possess a Lax pair by Chu and Scott [14], which describes the interaction of two laser beams, the pump wave (frequency ω_L) and the Stokes wave (down-shifted to ω_S) with a medium reduced to a collection of oscillators of frequency $\omega_L - \omega_S$. The boundary value problem

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for SRS which is integrable by IST is precisely the one which is physically relevant, namely the data of the input laser field values. This has allowed, for instance, to fully interpret the experiments of [15] of laser pulse propagation in H_2 gas and to understand the observed *Raman spike* [16].

In the meantime, remarkable progress have been performed in the construction of the spectral transform for boundary value problems, either for linear or integrable nonlinear evolutions [17].

Recently, two important extensions of coupled wave equations have been obtained. First the discrete integrable version of the SRS system have been constructed [18] on the basis of the Ablowitz–Ladik spectral problem. This was the first instance of an integrable discrete system having a singular (nonanalytic) dispersion relation. Second the (continuous) SRS system have been solved on the semi-line, that is with input laser fields boundary values in $x = 0$ [19]. The interesting result there is that the time evolution of the spectral transform (the reflection coefficient) allows for pole motion which implies that solitons can be created out of a medium initially at rest.

Our purpose here is to prove that the boundary value problem on the semi-line $n > 0$ for the discrete SRS system is solvable and to study the resulting consequences on the evolution of the spectrum.

Results

We solve the following discrete three-wave (A_1, A_2, q) system:

$$\begin{aligned} &A_1(\theta, n, t) - A_1(\theta, n-1, t) = \mathrm{e}^{\mathrm{i}n\theta} q(n, t)\, A_2(\theta, n, t) \\ &A_2(\theta, n, t) - A_2(\theta, n-1, t) = -\mathrm{e}^{-\mathrm{i}n\theta} \bar{q}(n, t)\, A_1(\theta, n, t) \\ &q_t(n, t) = -\frac{1}{2\pi} \int_{-\pi}^{+\pi} (A_1 * A_2)\, \mathrm{e}^{-\mathrm{i}n\theta}\, \mathrm{d}\theta. \end{aligned} \qquad (1.1)$$

The parameter θ varies in $[-\pi, +\pi]$, n is an integer and t the time. The interaction term here above is

$$(A_1 * A_2) = \frac{g\, A_1(n-1)\, \bar{A}_2(n) + \bar{g}\, A_1(n)\, \bar{A}_2(n-1)}{|A_1(n)|^2 + |A_2(n)|^2} \qquad (1.2)$$

where $g = g(\theta, t)$ is an arbitrary function in $L^2([-\pi, +\pi])$ which can also be time dependent.

One of the main results is the proof that the system (1.1) with data (*initial-boundary value problem*) on the semi line $n > 0$

$$q(n, 0) \qquad I_1(\theta, t) = A_1(\theta, 0, t) \qquad I_2(\theta, t) = A_2(\theta, 0, t) \qquad (1.3)$$

is solvable. To be more specific, the method gives the explicit output field values as $n \to \infty$

$$A_1(\infty) = \frac{|\tau|^2}{1 + |\rho|^2} \left(I_1 \frac{1}{\tau} - I_2 \frac{\rho}{\tau} \right) \qquad A_2(\infty) = \frac{|\tau|^2}{1 + |\rho|^2} \left(I_1 \frac{\bar{\rho}}{\bar{\tau}} + I_2 \frac{1}{\bar{\tau}} \right) \qquad (1.4)$$

in terms of the scattering data $\rho(\theta, t)$ (reflection coefficient) and $\tau(\theta, t)$ (transmission coefficient).

We shall demonstrate that ρ is constructed by solving the Riccati evolution

$$\rho_t = \nu_{12} + \rho(\nu_{11} - \nu_{22}) - \rho^2 \nu_{21} \qquad (1.5)$$

and τ by solving then the linear evolution

$$\frac{\tau_t}{\tau} = \mu_{22} - \nu_{22} - \rho \nu_{21} \qquad (1.6)$$

where the ν_{ij} and μ_{ij} are explicitly given from the input data in formulae (4.4) and (4.5).

In conclusion the output fields values (1.4) can be computed through a sequence of linear (or linearizable) operations which require only the input fields values I_1 and I_2 and the initial spectral datum $\rho(k, 0)$, spectral transform of the initial value $q(n, 0)$. The case when the medium is initially at rest, i.e. $q(n, 0) = 0$, corresponds to taking $\rho(k, 0) = 0$ and consequently the solution is explicit, which shall be illustrated on an example. In the case of an initial datum $q(n, 0)$ given on a finite interval, as the initial spectral transform $\rho(k, 0)$ can be explicitly computed [20], the solution is explicit too.

Lastly, we shall demonstrate that the field values $A_j(\theta, L, t)$ at any finite distance $n = L$ can be obtained from the scattering coefficients as the solution of an algebraic system of order L. This result is of great interest for numerical implementation of the IST method for the SRS equations, which will be the subject of forthcoming work.

Continuous limit

The very reason why the system (1.1) is called *discrete SRS* lies in the fact that its continuous limit is the model equation for stimulated Raman scattering. Indeed, under the new variables

$$x = n\epsilon \qquad 2\lambda = \frac{\theta}{\epsilon} \qquad a_j(\lambda, x, t) = A_j(\theta, n, t) \qquad Q(x, t) = \frac{q(n, t)}{\epsilon} \tag{1.7}$$

the first two equations of (1.1) become in the limit $\epsilon \to 0$

$$\partial_x a_1 = Q a_2 \mathrm{e}^{2\mathrm{i}\lambda x} \qquad \partial_x a_2 = -\bar{Q} a_1 \mathrm{e}^{-2\mathrm{i}\lambda x}. \tag{1.8}$$

A direct consequence of the above system is

$$\partial_x \left(|a_1|^2 + |a_2|^2\right) = 0 \tag{1.9}$$

and consequently this quantity can be calculated in $x = 0$, i.e.

$$|A_1|^2 + |A_2|^2 \to |a_1|^2 + |a_2|^2 = |I_1|^2 + |I_2|^2.$$

Now the time evolution of q becomes

$$Q_t = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \mathrm{d}\lambda \, \frac{g + \bar{g}}{|I_1|^2 + |I_2|^2} a_1 \bar{a}_2 \, \mathrm{e}^{-2\mathrm{i}\lambda x} \tag{1.10}$$

which finally reduces to SRS for

$$\frac{g + \bar{g}}{|I_1|^2 + |I_2|^2} = \pi g_0 \tag{1.11}$$

where g_0 is the Raman scattering coupling constant.

The system (1.8) and (1.10) (with the above choice) is actually a version of the well known SRS system as in [14], where the group velocity dispersion is not neglected [19]. The parameter λ then has the physical meaning of the spectral extension of the wavepackets of envelopes a_1 (pump) and a_2 (Stokes).

Then, the most natural application of the method developed here is the integrable discretization of the SRS equations, together with a simple algorithm for computing the nonlinear Fourier transform.

2. Construction of the integrable evolution

The Lax pair

Let us consider the following pair of operators:

$$\psi(k, n + 1) = U(n)\, \Lambda(k)\, \psi(k, n)\, \Lambda^{-1}(k) \tag{2.1}$$

$$\psi_t(k, n) = V(k, n)\, \psi(k, n) \tag{2.2}$$

where

$$U(n) = [1 - Q(n+1)]^{-1} \qquad Q(n) = \begin{pmatrix} 0 & q(n) \\ r(n) & 0 \end{pmatrix} \qquad \Lambda(k) = \begin{pmatrix} 1/z & 0 \\ 0 & z \end{pmatrix}$$

with $z^2 = k$. The compatibility between these two operators can be written as

$$U_t(n) = V(k, n+1)\, U(n) - U(n)\, \Lambda(k)\, V(k, n)\, \Lambda^{-1}(k) \tag{2.3}$$

or equivalently

$$Q_t(n+1) = U^{-1}(n)\, V(k, n+1) - \Lambda(k)\, V(k, n)\, \Lambda^{-1}(k)\, U^{-1}(n). \tag{2.4}$$

Particular choice for V

The general evolution (2.4) acquires interest when the k dependence is eliminated, which can be performed through the choice of the matrix $V(k, n, t)$ in terms of $U(k, n, t)$. For instance a polynomial dependence on k would lead to the Ablowitz–Ladik hierarchy. Here we choose

$$V(k, n) = \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}} \frac{\mathrm{d}\zeta}{\zeta - k} A(\zeta, n)\, G(\zeta)\, A^{-1}(\zeta, n) \begin{pmatrix} 1 & 0 \\ 0 & k/\zeta \end{pmatrix} \tag{2.5}$$

where $\mathcal{C}$ is the unit circle oriented anticlockwise, G is an arbitrary diagonal matrix

$$G(\zeta) = \begin{pmatrix} g_1(\zeta) & 0 \\ 0 & g_2(\zeta) \end{pmatrix} \tag{2.6}$$

and $A(\zeta, n)$ satisfies the spectral equation (2.1), i.e.

$$A(\zeta, n+1) = U(n)\, \Lambda(\zeta)\, A(\zeta, n)\, \Lambda^{-1}(\zeta). \tag{2.7}$$

The general evolution (2.4) now becomes

$$Q_t(n+1) = \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}} \frac{\mathrm{d}\zeta}{\zeta - k} M(k, \zeta, n) \tag{2.8}$$

with

$$\begin{aligned} M(k, \zeta, n) = {} & U^{-1}(n)\, A(\zeta, n+1)\, G(\zeta)\, A^{-1}(\zeta, n+1) \begin{pmatrix} 1 & 0 \\ 0 & k/\zeta \end{pmatrix} \\ & - \Lambda(k)\, A(\zeta, n)\, G(\zeta)\, A^{-1}(\zeta, n) \begin{pmatrix} 1 & 0 \\ 0 & k/\zeta \end{pmatrix} \Lambda^{-1}(k)\, U^{-1}(n). \end{aligned} \tag{2.9}$$

Using the identities

$$\begin{pmatrix} 1 & 0 \\ 0 & k/\zeta \end{pmatrix} \Lambda^{-1}(k) = \frac{z(k)}{z(\zeta)} \Lambda^{-1}(\zeta) \qquad \frac{z(k)}{z(\zeta)} \Lambda(k) = \begin{pmatrix} 1 & 0 \\ 0 & k/\zeta \end{pmatrix} \Lambda(\zeta)$$

together with the spectral equation (2.7), we finally obtain after some algebraic manipulation that the quantity $M(k, \zeta, n)/(k - \zeta)$ is actually k independent, and the evolution reads

$$Q_t(n+1) = -\frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}} \frac{\mathrm{d}\zeta}{2\zeta} \left[\sigma_3, \Lambda(\zeta)\, A(\zeta, n)\, G(\zeta)\, \Lambda^{-1}(\zeta)\, A^{-1}(\zeta, n+1)\right]. \tag{2.10}$$

Discrete SRS

To recover now from the above evolution the SRS system (1.1), we assume the reduction $r = -\bar{q}$. In that case, one can show that the matrices $A(\zeta, n)$ and $G(\zeta)$ have the following structures:

$$A = \begin{pmatrix} A_1 & \zeta^{-n}\bar{A}_2 \\ \zeta^n A_2 & -\bar{A}_1 \end{pmatrix} \qquad G = \begin{pmatrix} g & 0 \\ 0 & -\bar{g} \end{pmatrix}. \tag{2.11}$$

Equation (2.7) then results in

$$\begin{aligned} &A_1(\zeta, n, t) - A_1(\zeta, n-1, t) = \zeta^n q(n, t) A_2(\zeta, n, t) \\ &A_2(\zeta, n, t) - A_2(\zeta, n-1, t) = -\zeta^{-n}\bar{q}(n, t) A_1(\zeta, n, t) \end{aligned} \tag{2.12}$$

which is nothing but the first two equation of (1.1) for $\zeta = e^{i\theta}$. Next, it is only a matter of algebraic computation to reduce, with the structure (2.11), the evolution equation (2.10) to the form in (1.1).

Energy non-conservation

Note that in contrast to what occurs in the continuous case, the quantity

$$-\det\{A\} = |A_1|^2 + |A_2|^2 \tag{2.13}$$

is not a constant as indeed from (2.12) we deduce

$$|A_1(n+1)|^2 + |A_2(n+1)|^2 = \frac{|A_1(n)|^2 + |A_2(n)|^2}{1 + |q(n+1)|^2} \tag{2.14}$$

which shows that the total *energy flux density* is not conserved along the line, or that the light beams leave some energy on each visited site. After n steps,

$$|A_1(n+1)|^2 + |A_2(n+1)|^2 = \left(|I_1|^2 + |I_2|^2\right) \prod_{i=0}^{n+1} \frac{1}{1 + |q(i)|^2}. \tag{2.15}$$

Note on the dispersion relation

In the IST method, the principal spectral problem (2.1) is used to define a special solution, say $\varphi(k, n)$, by selecting particular boundary values in $n = 0$ (or $n \to \infty$). Then there exists a matrix $C(k, n)$ satisfying the equation

$$C(k, n+1) = \Lambda(k)\, C(k, n)\, \Lambda^{-1}(k) \tag{2.16}$$

such that

$$\varphi(k, n) = \psi(k, n)\, C(k, n). \tag{2.17}$$

Consequently the solution $\varphi(k, n)$ satisfies a modified version of the auxiliary spectral problem (2.2), namely

$$\varphi_t(k, n) = V(k, n)\, \varphi(k, n) + \varphi(k, n)\, \Omega(k, n) \tag{2.18}$$

where

$$\Omega(k, n) = C^{-1}(k, n)\, C_t(k, n). \tag{2.19}$$

The compatibility condition is not modified with that operator, as indeed we have

$$\Omega(k, n+1) = \Lambda(k)\, \Omega(k, n)\, \Lambda^{-1}(k) \tag{2.20}$$

and the matrix $\Omega(k, n)$ is called the *dispersion relation*.

3. The spectral problem on the semi-line

Jost solutions

Let us consider the Ablowitz–Ladik spectral problem (2.1)

$$\varphi(k,n{+}1)-\Lambda\,\varphi(k,n)\,\Lambda^{-1}=Q(n{+}1)\,\varphi(k,n{+}1)\qquad n\leqslant 0\quad\Rightarrow\quad Q(n)=0 \tag{3.1}$$

where k is the spectral parameter which belongs to the domain $\mathcal{D}=\mathbb{C}-\{0,\infty\}$. The solution of (3.1) possesses the property

$$\det\{\varphi(k,n)\}=\det\{\varphi(k,n{+}1)\}[1-r(n{+}1)q(n{+}1)]. \tag{3.2}$$

The solution of this spectral problem goes through the construction of some well chosen solutions (the Jost solutions) out of some particular asymptotic behaviours. These solutions are denoted by $\varphi^{\pm}$ and are defined by the following discrete integral equations ($n\geqslant 0$):

$$\begin{pmatrix}\varphi^+_{11}(k,n)\\ \varphi^+_{21}(k,n)\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}\displaystyle\sum_{i=1}^{n}q(i)\,\varphi^+_{21}(k,i)\\ \displaystyle\sum_{i=1}^{n}k^{n-i}r(i)\,\varphi^+_{11}(k,i)\end{pmatrix} \tag{3.3}$$

$$\begin{pmatrix}\varphi^+_{12}(k,n)\\ \varphi^+_{22}(k,n)\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}\displaystyle -\sum_{i=n+1}^{\infty}k^{i-n}q(i)\,\varphi^+_{22}(k,i)\\ \displaystyle\sum_{i=1}^{n}r(i)\,\varphi^+_{12}(k,i)\end{pmatrix} \tag{3.4}$$

$$\begin{pmatrix}\varphi^-_{11}(k,n)\\ \varphi^-_{21}(k,n)\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}\displaystyle\sum_{i=1}^{n}q(i)\,\varphi^-_{21}(k,i)\\ \displaystyle -\sum_{i=n+1}^{\infty}k^{n-i}r(i)\,\varphi^-_{11}(k,i)\end{pmatrix} \tag{3.5}$$

$$\begin{pmatrix}\varphi^-_{12}(k,n)\\ \varphi^-_{22}(k,n)\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}\displaystyle\sum_{i=1}^{n}k^{i-n}q(i)\,\varphi^-_{22}(k,i)\\ \displaystyle\sum_{i=1}^{n}r(i)\,\varphi^-_{12}(k,i)\end{pmatrix} \tag{3.6}$$

together with

$$n<0\quad\Rightarrow\quad\varphi^{\pm}(n)=\Lambda^{n}\varphi^{\pm}(0)\Lambda^{-n}. \tag{3.7}$$

Analytical properties

The column vectors $\varphi^+_1(n)$ is a polynomial of order $n-1$ in k and $\varphi^-_2(n)$ a polynomial of order $n-1$ in $1/k$. Hence $\varphi^+_1(n)$ can be analytically defined inside the unit circle and φ^-_2 outside. We prove in appendix A that the vector φ^+_2 is meromorphic inside the unit circle with a finite number N^+ of simple poles k^+_n (and φ^-_1 outside with N^- poles k^-_n).

These solutions obey the Riemann–Hilbert relations on the unit circle

$$\begin{aligned}\varphi^+_1-\varphi^-_1&=-\zeta^{n}\rho^{-}\varphi^-_2\\ \varphi^+_2-\varphi^-_2&=\zeta^{-n}\rho^{+}\varphi^+_1\end{aligned} \tag{3.8}$$

where the *reflection coefficients* are

$$\rho^+=-\sum_{i=1}^{\infty}\zeta^{i}q(i)\,\varphi^+_{22}(\zeta,i)\qquad\rho^-=-\sum_{i=1}^{\infty}\zeta^{-i}r(i)\,\varphi^-_{11}(\zeta,i). \tag{3.9}$$

Boundary behaviours and spectral coefficients

By direct observation we have

$$\begin{aligned} n=0:\quad & \varphi^+ = \begin{pmatrix} 1 & \rho^+ \\ 0 & 1 \end{pmatrix} \qquad \varphi^- = \begin{pmatrix} 1 & 0 \\ \rho^- & 1 \end{pmatrix} \\ n\to\infty:\quad & \varphi^+ \to \begin{pmatrix} \hat{\tau}^+ & 0 \\ \zeta^n \hat{\rho}^+ & \tau^+ \end{pmatrix} \qquad \varphi^- \to \begin{pmatrix} \tau^- & \zeta^{-n}\hat{\rho}^- \\ 0 & \hat{\tau}^- \end{pmatrix}. \end{aligned} \tag{3.10}$$

These bounds define also the transmission coefficients $\tau^\pm$ as

$$\tau^+ = 1 + \sum_{i=1}^{\infty} r(i)\,\varphi_{12}^+(\zeta,i) \qquad \tau^- = 1 + \sum_{i=1}^{\infty} q(i)\,\varphi_{21}^-(\zeta,i) \tag{3.11}$$

and the auxiliary spectral data $\hat{\rho}^\pm$ and $\hat{\tau}^\pm$ which are discussed in appendix B. Using (3.2) recursively from $n=0$ to ∞ we obtain the *unitarity relation*

$$1-\rho^+\rho^- = \tau^+\tau^- \prod_{i=1}^{\infty}[1-r(i)\,q(i)]. \tag{3.12}$$

Reduction

If $Q(n)$ satisfies the reduction $q(n) = -\bar{r}(n)$ one can verify directly from the discrete integral equations defining φ in (3.3)–(3.6) that

$$\sigma_2\overline{\varphi^+(1/\bar{k},n)}\,\sigma_2 = \varphi^-(k,n). \tag{3.13}$$

Then from the definitions (3.9), (3.11) it follows that for $|\zeta|=1$

$$\overline{\rho^+(\zeta)} = -\rho^-(\zeta) \qquad \overline{\tau^+(\zeta)} = \tau^-(\zeta) \tag{3.14}$$

and to simplify the formulae from now on we shall be using

$$\rho(\zeta) \equiv \rho^+(\zeta) \qquad \tau(\zeta) \equiv \tau^+(\zeta). \tag{3.15}$$

The unitarity relation (3.12) can be rewritten as

$$1+|\rho|^2 = |\tau|^2 F \tag{3.16}$$

where

$$F = \prod_{i=1}^{\infty}\left(1+|q(i)|^2\right). \tag{3.17}$$

4. The solution of discrete SRS

Time evolution of the spectral data

The tool to solve the boundary value problem (1.3) for the discrete SRS system (1.1) is completed now by calculating the time evolution of the spectral coefficients $\rho(\zeta,t)$ and $\tau(\zeta,t)$. Indeed, once these evolutions found, the output field values (1.4), are explicitly calculable. The question of the evaluation of the *potential* $q(n,t)$ goes through the solution of the inverse problem which will be discussed later.

The evolution of the reflection coefficient $\rho(\zeta,t)$ is obtained from the auxiliary Lax operator (2.2), or more precisely from the version (2.18) adapted to the Jost solutions $\varphi^\pm$, by just taking its value in $n=0$ and its limit value as $n\to\infty$. This calculation, while technical, presents no particular difficulty and we leave it to appendix C. Let us just remark that the procedure gives the value of the dispersion relation $\Omega^{\pm}$ and the time evolution of all spectral data ($\rho^{\pm}$, $\tau^{\pm}$ $\hat{\rho}^{\pm}$ and $\hat{\tau}^{\pm}$), and that it furnishes more equations than necessary. Hence we shall have to verify consistency, namely that some of the equations are redundant.

The result of the process detailed in appendix C is the following time evolution of the reflection coefficient:

$$\rho_t = V^{+}_{12}(0) + \rho\big[V^{+}_{11}(0) - V^{+}_{22}(0)\big] - \rho^2 V^{+}_{21}(0) \tag{4.1}$$

and of the transmission coefficient

$$\tau_t = V^{+}_{22}(\infty)\tau - \big[V^{+}_{22}(0) + \rho V^{+}_{21}(0)\big]\tau \tag{4.2}$$

where $V_{ij}(0)$ (respectively $V_{ij}(\infty)$) denotes the ij-component of the matrix $V(k,n,t)$ defined in (2.5) taken in $n = 0$ (respectively in $n \to \infty$).

These evolutions can be written in a more compact form as

$$\rho_t = \nu_{12} + \rho(\nu_{11} - \nu_{22}) - \rho^2\nu_{21} \tag{4.3}$$

where the ν_{ij} are given from the input data as the components of the matrix (we note $\mathrm{e}^{\mathrm{i}\theta} = k$)

$$\frac{1}{2\pi\mathrm{i}}\oint_{\mathcal{C}}\frac{\mathrm{d}\zeta}{\zeta - (1-0)k}\,\frac{1}{|I_1|^2 + |I_2|^2}\begin{pmatrix} g|I_1|^2 - \bar{g}|I_2|^2 & (g+\bar{g})I_1\bar{I}_2 k/\zeta \\ (g+\bar{g})\bar{I}_1 I_2 & \big(g|I_2|^2 - \bar{g}|I_1|^2\big)k/\zeta \end{pmatrix} \tag{4.4}$$

where $\mathcal{C}$ is the unit circle. For the transmission coefficient we have the linear evolution (1.6) for

$$\mu_{22} = \frac{1}{2\pi\mathrm{i}}\oint_{\mathcal{C}}\frac{\mathrm{d}\zeta}{\zeta - (1-0)k}\,\frac{k}{\zeta}\,\frac{g|I_1\bar{\rho} + I_2|^2 - \bar{g}|I_1 - \rho I_2|^2}{\big(1 + |\rho|^2\big)\big(|I_1|^2 + |I_2|^2\big)}. \tag{4.5}$$

It is worth noting that the *boundary value* data I_1 and I_2 of (1.3) completely determine the coefficients of the above evolutions, while the *initial* datum of $q(n,0)$ determines the initial values $\rho(k,0)$ and $\tau(k,0)$ through the solution of the direct spectral problem (as usual in the IST method). The interesting aspect of our system is that the physically relevant case is that of two waves entering a medium initially at rest, that is for $q(n,0) = 0$ corresponding to

$$\rho(k,0) = 0 \qquad \tau(k,0) = 1. \tag{4.6}$$

We note also that the evolutions (4.1) which furnishes the solution is constituted of an inhomogeneous term $V^{+}_{12}(0)$ responsible for growth of the medium excitation q, a linear factor ensuring Raman amplification and a nonlinear part (in ρ^2) allowing for pole motion in the complex plane.

Inverse problem

The IST method allows also to solve for the value of the medium excitation $q(n,t)$ for all n and any given time t. This is obtained by solving the *inverse problem*, namely the reconstruction of the Jost solutions, and of the *potential* q, from the data of $\rho(k,t)$ and $\tau(k,t)$. This is done as follows.

The potentials are obtained by inserting into (2.1) the asymptotic expansions at large k and at small k of, respectively φ^{-}_{2} and φ^{+}_{1}. The result is

$$q(n+1) = -\varphi^{+(1)}_{12}(n) \qquad r(n+1) = -\varphi^{-(-1)}_{21}(n) \tag{4.7}$$

where $\varphi^{+(1)}_{12}(n)$ is the coefficient of k in the Taylor expansion for $k \to 0$ of $\varphi^{+}_{12}(k,n)$ and $\varphi^{-(-1)}_{21}(n)$ the coefficient of $1/k$ in the Laurent expansion for $k \to \infty$ of $\varphi^{-}_{21}(k,n)$.

Note that from (3.5) and (3.4) we have

$$\lim_{k\to\infty} \varphi_1^-(k,n) = \binom{1}{0} \tag{4.8}$$

$$\lim_{k\to 0} \varphi_2^+(k,n) = \binom{0}{1}. \tag{4.9}$$

Taking into account these asymptotic behaviours and their analytical properties we deduce that the Jost solutions φ_1^- and φ_2^+ can be reconstructed by solving the following Cauchy–Green integral equations:

$$\begin{aligned}\begin{pmatrix} \varphi_{11}^-(k,n) \\ \varphi_{21}^-(k,n) \end{pmatrix} &= \binom{1}{0} - \frac{1}{2\pi \mathrm{i}} \oint_C \mathrm{d}\zeta \frac{\zeta^n \rho^-(\zeta)}{\zeta - (1+0)k} \begin{pmatrix} \varphi_{12}^-(\zeta,n) \\ \varphi_{22}^-(\zeta,n) \end{pmatrix} \\ &\quad - \sum_j (k_j^-)^n \operatorname*{Res}_{k_j^-}\{\rho^-\} \frac{1}{k_j^- - k} \begin{pmatrix} \varphi_{12}^-(k_j^-,n) \\ \varphi_{22}^-(k_j^-,n) \end{pmatrix}\end{aligned} \tag{4.10}$$

$$\begin{aligned}\begin{pmatrix} \varphi_{12}^+(k,n) \\ \varphi_{22}^+(k,n) \end{pmatrix} &= \binom{0}{1} + \frac{1}{2\pi \mathrm{i}} \oint_C \mathrm{d}\zeta \frac{\zeta^{-n} \rho^+(\zeta)}{\zeta - (1-0)k} \frac{k}{\zeta} \begin{pmatrix} \varphi_{11}^+(\zeta,n) \\ \varphi_{21}^+(\zeta,n) \end{pmatrix} \\ &\quad - \sum_j \left(k_j^+\right)^{-n} \operatorname*{Res}_{k_j^+}\{\rho^+\} \frac{k}{k_j^+} \frac{1}{k_j^+ - k} \begin{pmatrix} \varphi_{11}^+(k_j^+,n) \\ \varphi_{21}^+(k_j^+,n) \end{pmatrix}.\end{aligned} \tag{4.11}$$

By using (C.2) and the Riemann–Hilbert relations we deduce that

$$\begin{aligned}\begin{pmatrix} \varphi_{11}^+(k,n) \\ \varphi_{21}^+(k,n) \end{pmatrix} &= \binom{1}{0} - \frac{1}{2\pi \mathrm{i}} \oint_C \mathrm{d}\zeta \frac{\zeta^n \rho^-(\zeta)}{\zeta - (1-0)k} \begin{pmatrix} \varphi_{12}^-(\zeta,n) \\ \varphi_{22}^-(\zeta,n) \end{pmatrix} \\ &\quad - \sum_j (k_j^-)^n \operatorname*{Res}_{k_j^-}\{\rho^-\} \frac{1}{k_j^- - k} \begin{pmatrix} \varphi_{12}^-(k_j^-,n) \\ \varphi_{22}^-(k_j^-,n) \end{pmatrix}\end{aligned} \tag{4.12}$$

$$\begin{aligned}\begin{pmatrix} \varphi_{12}^-(k,n) \\ \varphi_{22}^-(k,n) \end{pmatrix} &= \binom{0}{1} + \frac{1}{2\pi \mathrm{i}} \oint_C \mathrm{d}\zeta \frac{\zeta^{-n} \rho^+(\zeta)}{\zeta - (1+0)k} \frac{k}{\zeta} \begin{pmatrix} \varphi_{11}^+(\zeta,n) \\ \varphi_{21}^+(\zeta,n) \end{pmatrix} \\ &\quad - \sum_j \left(k_j^+\right)^{-n} \operatorname*{Res}_{k_j^+}\{\rho^+\} \frac{k}{k_j^+} \frac{1}{k_j^+ - k} \begin{pmatrix} \varphi_{11}^+(k_j^+,n) \\ \varphi_{21}^+(k_j^+,n) \end{pmatrix}.\end{aligned} \tag{4.13}$$

Consistency of the evolution on the semi-line

The consistency of the method results in verifying that the reconstructed eigenfunction does obey the boundary condition (3.7). This is actually a consequence of the analytical properties of ρ at all t, which is a consequence of (4.1) where the coefficients are analytic functions inside the unit disc.

Indeed for $\rho^+(k)$ meromorphic inside the unit disc with a zero at $k = 0$ and for $\rho^-(k)$ meromorphic outside the unit disc with a zero at $k = \infty$ we have that

$$\varphi_1^+(k,n) = \binom{1}{0} \qquad \text{for} \quad n \leqslant 0 \tag{4.14}$$

$$\varphi_2^-(k,n) = \binom{0}{1} \qquad \text{for} \quad n \leqslant 0. \tag{4.15}$$

Inserting these values into (4.10) and (4.11) we obtain

$$\varphi_2^+(k,n) = \begin{pmatrix} k^{-n}\rho^+(k) \\ 1 \end{pmatrix} \qquad \text{for} \quad n \leqslant 0$$

$$\varphi_1^-(k,n) = \begin{pmatrix} 1 \\ k^n\rho^-(k) \end{pmatrix} \qquad \text{for} \quad n \leqslant 0.$$

Then thanks to (4.7) and recalling that $\rho^+(k)$ has a zero at $k = 0$ and $\rho^-(k)$ a zero at $k = \infty$ we get that the reconstructed potential $Q(n) = 0$ for $n \leqslant 0$.

Algebraic solution of the inverse problem

The inverse problem can be solved without using the Cauchy–Green integral equations but just using directly the Riemann–Hilbert relations (3.8), together with the information that the column vector $\varphi_1^+(n)$ is a polynomial of order $n-1$ in k and $\varphi_2^-(n)$ a polynomial of order $n-1$ in $1/k$, that the asymptotic behaviours of $\varphi_2^+(n)$ and $\varphi_1^-(n)$ at $k = 0$ and at $k = \infty$, respectively, are given by (4.9) and (4.8) and that $\rho^+(k)$ and $\rho^-(k)$ are meromorphic, respectively, inside and outside the unit disc with a zero at $k = 0$ and at $k = \infty$.

Using this information the spectral data are expressed as

$$\rho^+(k) = \sum_{i=1}^{\infty} s^+(i)\, k^i \tag{4.16}$$

$$\rho^-(k) = \sum_{i=1}^{\infty} s^-(i)\, k^{-i} \tag{4.17}$$

and we seek the solution of the inverse problem under the form

$$\varphi_1^+(k,n) = \sum_{i=0}^{n-1} d^+(i,n)\, k^i \tag{4.18}$$

$$\varphi_2^-(k,n) = \sum_{i=0}^{n-1} d^-(i,n)\, k^{-i}. \tag{4.19}$$

Solving the inverse problem is then understood as finding the unknowns 2-vectors $d^{\pm}(i,n)$ in terms of the (scalar) data $s^{\pm}(i)$.

The first and second equation of (3.8) can be extended analytically, respectively, outside and inside the unit disc. We insert (4.18), (4.19), (4.16) and (4.17) and impose that k^i and k^{-i} terms cancel out. Then we get $(m = 0, 1, \ldots, n-1)$ the following algebraic system of equations for the unknowns $d^{\pm}(i,n)$

$$\sum_{j=0}^{n-1} a_{mj} d^+(j,n) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} s^-(n-m) - \delta_{m0} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.20}$$

$$\sum_{j=0}^{n-1} b_{mj} d^-(j,n) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} s^+(n-m) - \delta_{m0} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.21}$$

where the coefficients are expressed in terms of the data as

$$a_{mj} = \sum_{i=0}^{n-j-1} s^+(n-j-i)s^-(n-m-i) \qquad \text{for} \quad j \leqslant m-1 \tag{4.22}$$

$$a_{mm} = \sum_{i=0}^{n-m-1} s^+(n-m-i)s^-(n-m-i) - 1 \tag{4.23}$$

$$a_{mj} = \sum_{i=0}^{n-m-1} s^+(n-j-i)s^-(n-m-i) \qquad \text{for} \quad j \geqslant m+1 \tag{4.24}$$

and

$$b_{mj} = \sum_{i=0}^{n-j-1} s^-(n-j-i)s^+(n-m-i) \qquad \text{for} \quad j \leqslant m-1 \tag{4.25}$$

$$b_{mm} = \sum_{i=0}^{n-j-1} s^-(n-j-i)s^+(n-m-i) - 1 \tag{4.26}$$

$$b_{mj} = \sum_{i=0}^{n-m-1} s^-(n-j-i)s^+(n-m-i) \qquad \text{for} \quad j \geqslant m+1. \tag{4.27}$$

By factorization of the coefficients of $1/k$ and k in (3.8), we obtain the expressions of $\varphi_1^{-(-1)}(n)$ and $\varphi_2^{+(1)}(n)$ which from (4.7) lead to the potentials

$$r(n+1) = -\sum_{i=0}^{n-1} d_2^-(i,n)s^-(n-i+1) \tag{4.28}$$

$$q(n+1) = -\sum_{i=0}^{n-1} d_1^+(i,n)s^+(n-i+1) \tag{4.29}$$

where the $d_j^{\pm}$'s denote the two components of the vector $d^{\pm}$.

In the case of the reduction $r = -\bar{q}$ we have

$$\overline{d_1^+(i,n)} = d_2^-(i,n) \tag{4.30}$$

$$\overline{d_2^+(i,n)} = -d_1^-(i,n) \tag{4.31}$$

$$\overline{s^+(i)} = -s^-(i). \tag{4.32}$$

In this case the matrices of the coefficients of the two systems of equations in the unknowns $d^+(j,n)$ and $d^+(j,n)$ are adjoints with diagonal elements $\leqslant 0$.

5. A significant example

Let us consider the case

$$I_2(k,t) = a\mathrm{e}^{\mathrm{i}\phi t} I_1(k,t) \tag{5.1}$$

where $I_1(k,t)$ is an arbitrary function and $\phi \in \mathbb{R}$ and $a \in \mathbb{C}$ are arbitrary constants. This choice corresponds to a Stokes envelope input $I_2(k,t)$ which is a portion of the pump input $I_1(k,t)$ and with a polarization direction rotating around that of $I_1(k,t)$ at angular velocity ϕ.

From (4.1) and (C.12), in the reduced case $q = -\bar{r}$, we obtain the following evolution equation for the reflection coefficient:

$$\rho_t + e^{i\phi t}\alpha\rho^2 + (\beta + i\phi)\rho + e^{-i\phi t}k\gamma = 0 \tag{5.2}$$

where

$$\alpha(k) = \frac{2a}{\left(1+|a|^2\right)}\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{d\zeta\; g_R(\zeta)}{(\zeta - (1-0)k)} \tag{5.3}$$

$$\beta(k) = -\left(\frac{1-|a|^2}{1+|a|^2}\right)\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{d\zeta\; g_R(\zeta)}{(\zeta - (1-0)k)}(1 + k/\zeta) - i\phi - i\phi_0 \tag{5.4}$$

$$\gamma(k) = -\frac{2\bar{a}}{\left(1+|a|^2\right)}\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{d\zeta\; g_R(\zeta)}{(\zeta - (1-0)k)\zeta} \tag{5.5}$$

$$\phi_0 = -\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{d\zeta\; g_I(\zeta)}{\zeta} \tag{5.6}$$

and $g_R(\zeta)$ and $g_I(\zeta)$ are the real and imaginary part of the coupling function, respectively. It is easy to show that changing $g_I(\zeta)$ is equivalent to change the phase ϕ in the equation (5.1) relating the two initial data I_1 and I_2. Therefore, without loss of generality, we can choose $g_I(\zeta) \equiv 0$ (and then $\phi_0 = 0$), while $g_R(\zeta)$ is left arbitrary.

The Riccati equation (5.2) can be solved explicitly giving

$$\rho(t) = e^{-i\phi t}\frac{\Gamma\rho_0 - (\beta\rho_0 + 2k\gamma)\tanh\Gamma t/2}{\Gamma + (2\alpha\rho_0 + \beta)\tanh\Gamma t/2} \tag{5.7}$$

where

$$\rho_0(k) = \rho(k, 0) \tag{5.8}$$

$$\Gamma(k) = \sqrt{\beta(k)^2 - 4k\alpha(k)\,\gamma(k)}. \tag{5.9}$$

Since ρ is invariant for the exchange $\Gamma \to -\Gamma$, it has no cut related to the square root necessary to obtain Γ. Consequently, thanks also to the analytical properties of α, β and γ, ρ has the same analytical properties of ρ_0 (that is meromorphic inside the unit disc with a zero at $k = 0$) with some possible additional poles due to the zeros of the denominator.

For the evolution equation of F we get from (C.18)

$$\frac{F_t}{F} = \frac{1}{\pi i}\oint_{\mathcal{C}}\frac{d\zeta\; g_R}{\zeta}\frac{|\rho|^2\left(1-|a|^2\right) + \bar{a}\bar{\rho}\,e^{-i\phi t} + a\rho\,e^{i\phi t}}{\left(1+|a|^2\right)\left(1+|\rho|^2\right)}. \tag{5.10}$$

In particular, if the initial values $q(n, 0)$ are different from zero only on a finite interval the spectral initial data ρ_0 can be explicitly computed via a finite step of algebraic operation (see [20]) and, then, once performed the integrals (5.3)–(5.5), one finds from (5.7) the spectral data $\rho(k, t)$ at time t. Finally, the initial value problem is solved by integrating (5.10) and by following the algebraic procedure described in section 4.

Appendix A. Analytical properties

The discrete integral equation (3.3) defining φ_1^+ can be rewritten as

$$\varphi_{11}^+(k, n)(1 - r(n)q(n)) = 1 + q(n)\sum_{i=1}^{n-1}k^{n-i}r(i)\,\varphi_{11}^+(k, i) + \sum_{i=1}^{n-1}q(i)\,\varphi_{21}^+(k, i)$$

$$\varphi_{21}^+(k, n) = r(n)\varphi_{11}^+(k, n) + \sum_{i=1}^{n-1}k^{n-i}r(i)\,\varphi_{11}^+(k, i)$$

which allows to prove by induction that $\varphi_1^+(n)$ is a polynomial of order $n-1$ in k. Analogously, using (3.6) one can show that $\varphi_1^-(n)$ is a polynomial of order $n-1$ in $1/k$.

The integral equation for φ_2^+, rewritten as

$$\begin{pmatrix} \varphi_{12}^+(k,n) \\ \varphi_{22}^+(k,n) \end{pmatrix} = \begin{pmatrix} 0 \\ \tau^+(k) \end{pmatrix} - \sum_{i=n+1}^{\infty} \begin{pmatrix} k^{i-n}\, q(i)\, \varphi_{22}^+(k,i) \\ r(i)\, \varphi_{12}^+(k,i) \end{pmatrix} \tag{A.1}$$

shows that the function $\psi_2^+ = \varphi_2^+/\tau^+$ satisfies the discrete Volterra-like integral equation

$$\begin{pmatrix} \psi_{12}^+(k,n) \\ \psi_{22}^+(k,n) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{i=n+1}^{\infty} \begin{pmatrix} k^{i-n}\, q(i)\, \psi_{22}^+(k,i) \\ r(i)\, \psi_{12}^+(k,i) \end{pmatrix}. \tag{A.2}$$

Hence ψ_2^+ is analytic inside the unit disc. Then, writing τ^+ as

$$\tau^+(k) = \left[1 - \sum_{i=1}^{\infty} r(i)\, \psi_{12}^+(k,i)\right]^{-1} \tag{A.3}$$

we deduce that $\tau^+(k)$ is meromorphic inside the unit disc with poles the zeros of $1 - \sum_{i=1}^{\infty} r(i)\, \psi_{12}^+(k,i)$. Consequently φ_2^+ is meromorphic inside the unit disc with poles at $k = k_j^+$ that we assume to be simple.

Note that from the first line of (A.1) and the definition of ρ^+ in (3.9) we have

$$\rho^+(k) = k^n\, \varphi_{12}^+(k,n) - \sum_{i=1}^{n} k^i\, q(i)\, \varphi_{22}^+(k,i). \tag{A.4}$$

Therefore $\rho^+(k)$, in contrast with the full line case, can be continued analytically inside the unit disc where it has the same analytical properties as $\varphi_2^+(k)$. Note also that $\rho^+(k)$ has a zero at $k=0$.

From (A.4) and the second line of (3.4) after multiplication by $k - k_j^+$ we get in the limit $k \to k_j^+$

$$\left(k_j^+\right)^n \operatorname*{Res}_{k_j^+} \varphi_{12}^+(k,n) = C_j^+ + \sum_{i=1}^{n} \left(k_j^+\right)^i q(i) \operatorname*{Res}_{k_j^+} \varphi_{22}^+(k,i)$$

$$\operatorname*{Res}_{k_j^+} \varphi_{22}^+(k,n) = \sum_{i=1}^{n} r(i) \operatorname*{Res}_{k_j^+} \varphi_{12}^+(k,n)$$

with

$$C_j^+ = \operatorname*{Res}_{k_j^+} \rho^+(k). \tag{A.5}$$

Comparing with (3.3) we have

$$\operatorname*{Res}_{k_j^+} \varphi_2^+(k,n) = \left(k_j^+\right)^{-n} C_j^+ \varphi_1^+\left(k_j^+, n\right). \tag{A.6}$$

Analogously we can show that φ_1^- is meromorphic outside the unit disc with simple poles at $k = k_j^-$, that $\rho^-(k)$ has the same analytical properties, etc.

The analytical properties of φ can be summarized by writing

$$\frac{\partial \varphi(k,n)}{\partial \bar{k}} = \varphi(k,n)\, R(k,n) \tag{A.7}$$

where

$$R(k,n) = \begin{pmatrix} 0 & \rho^+(k)\delta^+(k,1)k^{-n} \\ -\rho^-(k)\delta^-(k,1)k^n & \end{pmatrix}$$
$$-2\pi \mathrm{i} \begin{pmatrix} 0 & \sum_{j=1}^{N^+} C_j^+ \delta\left(k - k_j^+\right) k^{-n} \\ \sum_{j=1}^{N^-} C_j^- \delta(k - k_j^-) k^n & 0 \end{pmatrix}. \tag{A.8}$$

The distributions $\delta^{\pm}(k,1)$ have support on the unit circle $\mathcal{C}$ in the complex k-plane and are defined by the following formula:

$$\iint \mathrm{d}k \wedge \mathrm{d}\bar{k}\, \delta^{\pm}(k,1)\, f(k) = \oint_{\mathcal{C}} \mathrm{d}\zeta\ f((1 \mp 0)\zeta). \tag{A.9}$$

The distributions $\delta\big(k - k_j^{\pm}\big)$ have support on the point $k = k_j^{\pm}$ and are defined by

$$\iint \mathrm{d}k \wedge \mathrm{d}\bar{k}\, \delta\big(k - k_j^{\pm}\big) f(k) = f\big(k_j^{\pm}\big). \tag{A.10}$$

In the reduced case $q(n) = -\overline{r(n)}$ taking into account the reduction property for $\rho^{\pm}$ in (3.14) and the definitions (A.5), we have

$$k_j^{+} = \frac{1}{\bar{k}_j^{-}} \qquad \frac{C_j^{+}}{k_j^{+}} = \overline{\left(\frac{C_j^{-}}{k_j^{-}}\right)}. \tag{A.11}$$

Appendix B. Auxiliary spectral data

The auxiliary spectral data used in the boundary behaviours (3.10) are defined as

$$\hat{\rho}^{+} = \sum_{i=1}^{\infty} \zeta^{-i}\, r(i)\, \varphi_{11}^{+}(\zeta, i) \qquad \hat{\rho}^{-} = \sum_{i=1}^{\infty} \zeta^{i}\, q(i)\, \varphi_{22}^{-}(\zeta, i) \tag{B.1}$$

$$\hat{\tau}^{+} = 1 + \sum_{i=1}^{\infty} q(i)\, \varphi_{21}^{+}(\zeta, i) \qquad \hat{\tau}^{-} = 1 + \sum_{i=1}^{\infty} r(i)\, \varphi_{12}^{-}(\zeta, i). \tag{B.2}$$

By computing $\hat{\rho}^{\pm} + \rho^{\mp}$ and using the R–H relations (3.8) we readily obtain

$$\hat{\rho}^{+} = -\rho^{-}\hat{\tau}^{-} \qquad \hat{\rho}^{-} = -\rho^{+}\hat{\tau}^{+} \tag{B.3}$$

and similarly from $\hat{\tau}^{\pm} - \tau^{\mp}$ we obtain

$$\hat{\tau}^{+} = \frac{\tau^{-}}{1 - \rho^{+}\rho^{-}} \qquad \hat{\tau}^{-} = \frac{\tau^{+}}{1 - \rho^{+}\rho^{-}} \tag{B.4}$$

which, in turn, imply

$$\hat{\rho}^{+} = -\frac{\rho^{-}\tau^{+}}{1 - \rho^{+}\rho^{-}} \qquad \hat{\rho}^{-} = -\frac{\rho^{+}\tau^{-}}{1 - \rho^{+}\rho^{-}}. \tag{B.5}$$

Finally, formulae (B.4) and (B.5) in the reduced case $r = -\bar{q}$, can be rewritten by means of (3.12) and (3.17) as

$$\hat{\tau}^{+} = \frac{1}{\tau F} \qquad \hat{\tau}^{-} = \frac{1}{\bar{\tau} F} \tag{B.6}$$

$$\hat{\rho}^{+} = \frac{\bar{\rho}}{\bar{\tau} F} \qquad \hat{\rho}^{-} = -\frac{\rho}{\tau F}. \tag{B.7}$$

Appendix C. Evolution of the spectral data

For deriving the evolution equation for the spectral data we use the evolution equation for φ^{+} in (2.18) that we write successively in $n = 0$ and $n = \infty$.

Let us recall first that

$$\lim_{n\to\pm\infty} \frac{1}{2\pi\mathrm{i}} \oint_{\mathcal{C}} \frac{\mathrm{d}\zeta}{\zeta - k} \left(\frac{k}{\zeta}\right)^{n} \Phi(\zeta) = \mp\frac{1}{2}\Phi(k) \qquad |k| = 1 \tag{C.1}$$

$$\oint_{\mathcal{C}} \frac{\mathrm{d}\zeta}{\zeta - (1 \mp 0)k} f(\zeta) = \pm\mathrm{i}\pi f(k) + P \oint_{\mathcal{C}} \frac{\mathrm{d}\zeta}{\zeta - k} f(\zeta) \qquad |k| = 1. \tag{C.2}$$

We rewrite (2.18) as

$$\Lambda^{-n}(k)\,\varphi_t^+(k,n)\,\Lambda^n(k) = \Lambda^{-n}(k)\,V^+(k,n)\,\Lambda^n(k)\,\Lambda^{-n}(k)\,\varphi^+(k,n)\,\Lambda^n(k) + \Lambda^{-n}(k)\,\varphi^+(k,n)\,\Lambda^n(k)\,\omega^+(k) \qquad (C.3)$$

where we introduce, for convenience, the matrix $\omega(k) = \Lambda^{-n}(k)\,\Omega(k)\,\Lambda^n(k)$, and compute the behaviour as $n \to \infty$ of the matrix $\Lambda^{-n}(k)\,V^+(k,n)\,\Lambda^n(k)$ which, using (C.1) and (C.2), results in

$$\lim_{n\to\infty}\left\{\Lambda^{-n}(k)\,V^+(k,n)\,\Lambda^n(k)\right\} = \begin{pmatrix} v_{11}^+(k,\infty) & 0 \\ v_{21}^+(k,\infty) & v_{22}^+(k,\infty) \end{pmatrix} \qquad (C.4)$$

with

$$v_{11}^+(k,\infty) = \frac{1}{2\pi \mathrm{i}}\oint_{\mathcal{C}} \frac{\mathrm{d}\zeta}{\zeta-(1-0)k}\,\frac{g|I_1-I_2\rho|^2-\bar{g}|I_1\bar{\rho}+I_2|^2}{\left(1+|\rho|^2\right)\left(|I_1|^2+|I_2|^2\right)}$$

$$v_{22}^+(k,\infty) = -\frac{1}{2\pi \mathrm{i}}\oint_{\mathcal{C}} \frac{\mathrm{d}\zeta}{\zeta-(1-0)k}\,\frac{\bar{g}|I_1-I_2\rho|^2-g|I_1\bar{\rho}+I_2|^2}{\left(1+|\rho|^2\right)\left(|I_1|^2+|I_2|^2\right)}\,\frac{k}{\zeta}$$

$$v_{21}^+(k,\infty) = \frac{(g+\bar{g})\tau\overline{(I_1-I_2\rho)}(I_1\bar{\rho}+I_2)}{\bar{\tau}\left(1+|\rho|^2\right)\left(|I_1|^2+|I_2|^2\right)}.$$

Therefore in the limit $n \to \infty$ from the element 12 of (C.3) we find $\omega_{12}^+ = 0$, and from the other matrix elements

$$\tau_t = v_{22}^+(\infty)\tau + \omega_{22}^+\tau \qquad (C.5)$$

$$\hat{\tau}_t = v_{11}^+(\infty)\hat{\tau} + \omega_{11}^+\hat{\tau} \qquad (C.6)$$

$$\hat{\rho}_t = v_{21}^+(\infty)\hat{\tau} + \left(v_{22}^+(\infty)+\omega_{11}^+\right)\hat{\rho} + \omega_{21}^+\tau. \qquad (C.7)$$

Evaluating the same evolution equation at $n = 0$ we have

$$0 = V_{11}^+(0) + \omega_{11}^+ + \rho\omega_{21}^+ \qquad (C.8)$$

$$0 = V_{21}^+(0) + \omega_{21}^+ \qquad (C.9)$$

$$0 = V_{22}^+(0) + \omega_{22}^+ + \rho V_{21}^+(0) \qquad (C.10)$$

$$\rho_t = V_{12}^+(0) + \rho V_{11}^+(0) + \rho\omega_{22}^+ \qquad (C.11)$$

where

$$V^+(k,0) = \frac{1}{2\pi \mathrm{i}}\oint_{\mathcal{C}} \frac{\mathrm{d}\zeta}{(\zeta-(1-0)k)\left(|I_1|^2+|I_2|^2\right)} \times \begin{pmatrix} g|I_1|^2-\bar{g}|I_2|^2 & (g+\bar{g})I_1\bar{I}_2 \\ (g+\bar{g})\bar{I}_1 I_2 & g|I_2|^2-\bar{g}|I_1|^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & k/\zeta \end{pmatrix}. \qquad (C.12)$$

We therefore obtain for the evolution equation (4.1) of ρ and

$$\omega^+ = \begin{pmatrix} -V_{11}^+(0)+\rho V_{21}^+(0) & 0 \\ -V_{21}^+(0) & -V_{22}^+(0)-\rho V_{21}^+(0) \end{pmatrix}. \qquad (C.13)$$

Inserting the value obtained for ω into (C.5)–(C.7) we get evolution equation (4.2) together with the following evolutions of the auxiliary spectral data:

$$\hat{\tau}_t = v_{11}^+(\infty)\hat{\tau} - \left(V_{11}^+(0)-\rho V_{21}^+(0)\right)\hat{\tau} \qquad (C.14)$$

$$\hat{\rho}_t = v_{21}^+(\infty)\hat{\tau} + \left(v_{22}^+(\infty)-V_{11}^+(0)+\rho V_{21}^+(0)\right)\hat{\rho} - V_{21}^+(0)\tau. \qquad (C.15)$$

Then by using (B.6) we have

$$\hat{\tau}_t = -\frac{\tau_t}{\tau}\hat{\tau} - \frac{F_t}{F}\hat{\tau}. \qquad (C.16)$$

Inserting it into (4.2) we find

$$\frac{F_t}{F} = -v_{11}^{+}(\infty) - v_{22}^{+}(\infty) + V_{11}^{+}(0) + V_{22}^{+}(0) \tag{C.17}$$

or otherwise

$$\frac{F_t}{F} = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{\mathrm{d}\zeta}{\zeta}(g+\bar{g})\frac{|\rho|^2\left(|I_1|^2 - |I_2|^2\right) + I_1\bar{I}_2\bar{\rho} + \bar{I}_1 I_2 \rho}{\left(|I_1|^2 + |I_2|^2\right)\left(1+|\rho|^2\right)}. \tag{C.18}$$

On can verify consistency by checking that (C.15) is consequence of the other evolution equations.

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